Diferenciální počet (derivace)

Důkazy derivačních vzorců

Vzorce:

1. $(c)' = 0,\ c \in \mathbb{R}$
2. $(x^n)' = n \cdot x^{n-1}$
3. $(a^x)' = a^x \cdot \ln a$
4. $(\mathrm{e}^x)' = \mathrm{e}^x$
5. $(\log_a x)' = \dfrac{1}{x \cdot \ln a}$
6. $(\ln x)' = \dfrac{1}{x}$
7. $(\sin x)' = \cos x$
8. $(\cos x)' = -\sin x$
9. $(\operatorname{tg} x)' = \dfrac{1}{\cos^2 x}$
10. $(\operatorname{cotg} x)' = -\dfrac{1}{\sin^2 x}$
11. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$
12. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$
13. $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$
14. $(\operatorname{arccotg} x)' = -\dfrac{1}{1+x^2}$

Důkaz. (1)

$$(c)' = \lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} \right) = \lim_{h \to 0} \left(\frac{c-c}{h} \right) = \lim_{h \to 0} (0) = 0$$ □

Důkaz. (2)

$$(x_0^n)' = \lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} \right) = \lim_{h \to 0} \left(\frac{(x_0+h)^n - x_0^n}{h} \right) =$$

$$= \lim_{h \to 0} \left(\frac{x_0^n + n \cdot x_0^{n-1} \cdot h + ... + n \cdot x_0 \cdot h^{n-1} + h^n - x_0^n}{h} \right) =$$

$$= \lim_{h \to 0} \left(\frac{n \cdot x_0^{n-1} \cdot h + ... + n \cdot x_0 \cdot h^{n-1} + h^n}{h} \right) =$$

$$= \lim_{h \to 0} \left(\frac{h \left(n \cdot x_0^{n-1} + ... + n \cdot x_0 \cdot h^{n-2} + h^{n-1} \right)}{h} \right) =$$

$$= \lim_{h \to 0} \left(n \cdot x_0^{n-1} + ... + n \cdot x_0 \cdot h^{n-2} + h^{n-1} \right) = n \cdot x_0^{n-1} + 0 + ... + 0 + 0 = n \cdot x_0^{n-1}$$ □

Důkaz. (3)

$(a^{x_0})' = \left(e^{\ln(a^{x_0})} \right)' = \left(e^{x_0 \cdot \ln(a)} \right)'$

nyní provedeme substituci a obdržíme:

$u = x_0 \cdot \ln(a)$

tedy můžeme dosadit:

$(e^u)' = e^u \cdot (u)'$

po zpětném dosazení obdržíme:

$e^{x_0 \cdot \ln(a)} \cdot (x_0 \cdot \ln(a))' = e^{ln(a^{x_0})} \cdot \ln(a) = a^{x_0} \cdot \ln(a)$ □

Důkaz. (4)

$$(e^{x_0})' = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right) = \lim_{h\to 0}\left(\frac{e^{x_0+h}-e^{x_0}}{h}\right) = \lim_{h\to 0}\left(\frac{e^{x_0}\cdot e^h-e^{x_0}}{h}\right) =$$

$$= \lim_{h\to 0}\left(\frac{e^{x_0}\cdot(e^h-1)}{h}\right) = \lim_{h\to 0}(e^{x_0})\cdot\lim_{h\to 0}\left(\frac{(e^h-1)}{h}\right) = e^{x_0}\cdot 1 = e^{x_0}$$

□

Důkaz. (5)

$$(\log_a(x))' = \left(\frac{\ln(x)}{\ln(a)}\right)' = \frac{\frac{1}{x}}{\ln(a)} = \frac{1}{x\cdot\ln(a)}$$

□

Důkaz. (6)

$$(\ln(x))' = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right) = \lim_{h\to 0}\left(\frac{\ln(x_0+h)-\ln(x_0)}{h}\right) = \lim_{h\to 0}\left(\frac{\ln\left(\frac{x_0+h}{x_0}\right)}{h}\right) =$$

$$\lim_{h\to 0}\left(\frac{\ln\left(1+\frac{h}{x_0}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{1}{h}\ln\left(1+\frac{h}{x_0}\right)\right) = \lim_{h\to 0}\left(ln\left(1+\frac{h}{x_0}\right)^{\frac{1}{h}}\right)$$

nyní provedeme substituci a obdržíme:

$$t = \frac{h}{x_0}$$

$$t\cdot x_0 = h$$

$$\frac{1}{h} = \frac{1}{t\cdot x_0}$$

tedy můžeme dosadit:

$$\lim_{h\to 0}\left(\ln\left(1+\frac{h}{x_0}\right)^{\frac{1}{h}}\right) = \lim_{h\to 0}\left(\ln(1+t)^{\frac{1}{t.x_0}}\right) = \lim_{h\to 0}\left(\frac{1}{x_0}\ln(1+t)^{\frac{1}{t}}\right) =$$

$$= \frac{1}{x_0}\ln\left(\lim_{h\to 0}\left((1+t)^{\frac{1}{t}}\right)\right) = \frac{1}{x_0}\ln(e) = \frac{1}{x_0}$$

□

Důkaz. (7)

$$(\sin(x_0))' = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right) = \lim_{h\to 0}\left(\frac{\sin(x_0+h)-\sin(x_0)}{h}\right) =$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(\frac{x_0+h+x_0}{2}\right)\sin\left(\frac{x_0+h-x_0}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{2\cos\left(\frac{2x_0+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right) =$$

$$= \lim_{h\to 0}\left(\frac{\cos\left(\frac{2x_0+h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right) = \lim_{h\to 0}\left(\cos\left(\frac{2x_0+h}{2}\right)\right)\cdot\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right) =$$

$$= \cos\left(\frac{2x_0+0}{2}\right) = \cos(x_0)$$

□

Důkaz. (8)

$$(\cos(x_0))' = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right) = \lim_{h\to 0}\left(\frac{\cos(x_0+h)-\cos(x_0)}{h}\right) =$$

$$= \lim_{h\to 0}\left(\frac{-2\sin\left(\frac{x_0+h+x_0}{2}\right)\sin\left(\frac{x_0+h-x_0}{2}\right)}{h}\right) = \lim_{h\to 0}\left(-\frac{2\sin\left(\frac{2x_0+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right) =$$

$$= \lim_{h\to 0}\left(-\frac{\sin\left(\frac{2x_0+h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right) = \lim_{h\to 0}\left(-\sin\left(\frac{2x_0+h}{2}\right)\right)\cdot\lim_{h\to 0}\left(-\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right) =$$

$$= -\sin\left(\frac{2x_0+0}{2}\right) = -\sin(x_0) \qquad \square$$

Důkaz. (9)

$$(\mathrm{tg}(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{(\sin(x))'\cdot\cos(x)-\sin(x)\cdot(\cos(x))'}{(\cos(x))^2} = \frac{\cos(x)\cdot\cos(x)-\sin(x)\cdot(-\sin(x))}{\cos^2(x)} =$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \qquad \square$$

Důkaz. (10)

$$(\mathrm{cotg}(x))' = \left(\frac{\cos(x)}{\sin(x)}\right)' = \frac{(\cos(x))'\cdot\sin(x)-\cos(x)\cdot(\sin(x))'}{(\sin(x))^2} = \frac{(-\sin(x))\cdot\sin(x)-\cos(x)\cdot\cos(x)}{\sin^2(x)} =$$

$$= \frac{-\sin^2(x)-\cos^2(x)}{\sin^2(x)} = \frac{-\left(\sin^2(x)+\cos^2(x)\right)}{\sin^2(x)} = -\frac{1}{\sin^2(x)} \qquad \square$$

Důkaz. (11)

$y = \arcsin(x)$

pak platí:

$\sin(y) = x$

pokud se pustíme do derivace podle x, obdržíme:

$(\sin(y))' = (x)'$

$\cos(y)\,(y)' = 1$

$(y)' = \frac{1}{\cos(y)}$

Víme že platí:

$\sin^2(y)+\cos^2(y) = 1$

$\cos^2(y) = 1-\sin^2(y)$

$\cos(y) = \sqrt{1-\sin^2(y)}$

Proto platí:

$$(y)' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}} \qquad \square$$

Důkaz. (12)
$y = \arccos(x)$
pak platí:
$\cos(y) = x$
pokud se pustíme do derivace podle x, obdržíme:
$(\cos(y))' = (x)'$
$-\sin(y)\,(y)' = 1$
$(y)' = \dfrac{1}{-\sin(y)}$
Víme že platí:
$\sin^2(y) + \cos^2(y) = 1$
$\sin^2(y) = 1 - \cos^2(y)$
$\sin(y) = \sqrt{1 - \cos^2(y)}$
Proto platí:
$(y)' = \dfrac{1}{-\sin(y)} = -\dfrac{1}{\sqrt{1-\cos^2(y)}} = -\dfrac{1}{\sqrt{1-x^2}}$ □

Důkaz. (13)
$y = \mathrm{arctg}(x)$
pak platí:
$\mathrm{tg}(y) = x$
pokud se pustíme do derivace podle x, obdržíme:
$(\mathrm{tg}(y))' = (x)'$
$\dfrac{1}{\cos^2(y)}\,(y)' = 1$

$$(y)' = \cos^2(y) = \frac{\cos^2(y)}{1} = \frac{\cos^2(y)}{\cos^2(y)+\sin^2(y)} \cdot \frac{\frac{1}{\cos^2(y)}}{\frac{1}{\cos^2(y)}} = \frac{\frac{\cos^2(y)}{\cos^2(y)}}{\frac{\cos^2(y)}{\cos^2(y)} + \frac{\sin^2(y)}{\cos^2(y)}} = \frac{1}{1+\left(\frac{\sin(y)}{\cos(y)}\right)^2} =$$

$$\frac{1}{1+\mathrm{tg}^2(y)} = \frac{1}{1+x^2}$$ □

Důkaz. (14)
$y = \mathrm{arccotg}(x)$
pak platí:
$\mathrm{cotg}(y) = x$
pokud se pustíme do derivace podle x, obdržíme:
$(\mathrm{cotg}(y))' = (x)'$
$-\dfrac{1}{\sin^2(y)}\,(y)' = 1$

$$(y)' = -\sin^2(y) = -\frac{\sin^2(y)}{1} = -\frac{\sin^2(y)}{\sin^2(y)+\cos^2(y)} \cdot \frac{\frac{1}{\sin^2(y)}}{\frac{1}{\sin^2(y)}} = -\frac{\frac{\sin^2(y)}{\sin^2(y)}}{\frac{\sin^2(y)}{\sin^2(y)} + \frac{\cos^2(y)}{\sin^2(y)}} =$$

$$= -\frac{1}{1+\left(\frac{\cos(y)}{\sin(y)}\right)^2} = -\frac{1}{1+\mathrm{cotg}^2(y)} = -\frac{1}{1+x^2}$$ □